

A decomposition formula for the Kazhdan-Lusztig basis of affine Hecke algebras of rank 2

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Abstract

In this paper, we prove a decomposition formula for the Kazhdan-Lusztig basis of affine Hecke algebras of rank 2 with positive weight function. Then we discuss some applications of this kind of decomposition to Lusztig's conjectures P1-P15.

1 Introduction

1.1 For each Coxeter group, we have left cells, right cells, and two-sided cells, defined by Kazhdan-Lusztig basis of the corresponding Hecke algebra, see [KL79]. The cells of an affine Weyl group are studied in [Lus85, Lus87a, Lus87b, Lus89]. It is shown that the cells of affine Weyl groups have many remarkable properties and play an important role in representations of affine Hecke algebras. This theory about two-sided cells is also discussed by [Lus03] in the context of Coxeter groups with unequal parameters. Some important properties of two-sided cells are contained in the conjectures P1-P15, see [Lus03, §14]. In general, these conjectures are still open for Coxeter groups with unequal parameters, and even for finite Weyl group. One of our motivation is to try to understand these conjectures for affine Weyl groups of rank 2. However, we do not obtain a complete proof of P1-P15 in this case. The main result of this paper is a kind of decomposition formula for Kazhdan-Lusztig basis (Theorem 4.7), which will reduce P1-P15 in our case to proving P8 and determining Lusztig's \mathbf{a} -function explicitly.

Conjectures P1-P15 have been proved in the case of Coxeter groups with equal parameter, as a corollary of the positivity conjecture, see [EW14] and [Lus14]. In this paper, unless otherwise specified, P1-P15 refer to these conjectures for unequal parameters. It is known that positivity does not hold for Coxeter groups with unequal parameters, which makes P1-P15 still open. For finite Coxeter groups that admit unequal parameters (type B_n ($n \geq 3$), $I(m)$ (m even) and F_4), we have already known that P1-P15 hold for type $I(m)$, F_4 and for type B_n with “asymptotic parameters”, see [Gec11, Thm.5.3] and the references therein.

For an affine Weyl group, the author has proved that P1-P15 hold on a distinguished two-sided cell—the lowest two-sided cell, see [Xie15]. The main tool for this is a decomposition formula of Kazhdan-Lusztig basis related

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to the lowest two-sided cell. To generalize the method used in [Xie15], we are motivated to investigate the case of affine Weyl groups of rank 2 and focus on type \tilde{B}_2 and \tilde{G}_2 . The partitions of two-sided cells of an affine Weyl group of type \tilde{B}_2 and \tilde{G}_2 for various parameters have been worked out in [Gui08, Gui10], which provide us with a lot of nontrivial examples of cells and make it possible to check some properties directly by calculation. This is the reason why we are concerned with affine Weyl groups of rank 2. As the case of the lowest two-sided cell, our first step is to establish a decomposition formula for Kazhdan-Lusztig basis of affine Weyl groups of rank 2.

1.2 Let \mathbf{c} be a two-sided cell of an affine Weyl group W of type \tilde{B}_2 and \tilde{G}_2 . Observation from Guilhot's partitions on two-sided cells shows that \mathbf{c} can be written as the form

$$(1.1) \quad \mathbf{c} = \bigsqcup_{d \in D} B_d d U_d$$

where B_d, U_d are subsets of W , indexed by the elements $d \in D$, each of which is an involution in some finite parabolic subgroup. And the right cells in \mathbf{c} are all of the form $\Phi_{b,d} = b d U_d$ with $b \in B_d, d \in D$.

Let $\mathcal{H}_{<\mathbf{c}}$ be the two-sided ideal of the Hecke algebra \mathcal{H} spanned by the Kazhdan-Lusztig basis $C_z, z < \mathbf{c}$, where “ $z < \mathbf{c}$ ” means that z is in some two-sided cell \mathbf{c}' with $\mathbf{c}' < \mathbf{c}$ in the natural order on the set of two-sided cells. Then the main result of this paper can be stated as follows

$$(1.2) \quad C_{bdu} = E_b C_d F_u \mod \mathcal{H}_{<\mathbf{c}}$$

for all $d \in D, b \in B_d, u \in U_d$, where E_b and F_u are determined by equations $C_{bd} = E_b C_d \mod \mathcal{H}_{<\mathbf{c}}, C_{du} = C_d F_u \mod \mathcal{H}_{<\mathbf{c}}$.

As a corollary, the \mathcal{H} -bimodule $\mathcal{H}_{\leq \mathbf{c}}/\mathcal{H}_{<\mathbf{c}}$ (in the obvious meaning) is generated by $C_d, d \in D$. But a more important feature of this decomposition formula is that E_b and F_u are independent of each other. We will see that this feature is crucial for its application to conjectures P1-P15.

Assuming P4 and a weak form P8' of P8, and assuming $\mathbf{a}(d) = \deg h_{d,d,d}$ for $d \in D$, we can deduce P1-P15 for \tilde{B}_2 and \tilde{G}_2 . The main tool is the decomposition formula. And some easy calculations are needed. If the \mathbf{a} -value of each element of affine Weyl groups of type \tilde{B}_2 and \tilde{G}_2 is known, then we can know immediately the validity of P4 and of the assumption $\mathbf{a}(d) = \deg h_{d,d,d}$. Thus computing \mathbf{a} -functions of \tilde{B}_2 and \tilde{G}_2 (with unequal parameters) is interesting and basic. But we now have no method for it.

Note that the form (1.1) in general does not hold for any Coxeter group. One can check that the Weyl group of type B_3 with equal parameter has given a counterexample. So it is a little surprise that (1.1) holds for all the two-sided cells of affine Weyl groups of rank 2.

The results in this paper can be proved for affine Weyl groups of type A_2 easily in a similar way. So actually the affine Weyl group of rank 2 in this article are used to refer to affine Weyl groups of type \tilde{B}_2 and \tilde{G}_2 .

1.3 The organization of this paper is as follows. In Section 2, we recall some basic notions about Hecke algebras and fix some notations. In Section 3, we recall Guilhot's partition of affine Weyl groups of rank 2 into two-sided cells and summarize some observations for latter use. In Section 4, we prove the decomposition formula under some assumptions, see Theorem 4.7. In

Section 5, we prove these assumptions are satisfied by affine Weyl groups of rank 2, and hence the decomposition formula holds in these cases. In Section 6, we apply the decomposition formula to conjectures P1-P15 (see Theorem 6.1, 6.2), and then reduce P1-P15 of affine Weyl groups of rank 2 to P8' and computing Lusztig's \mathbf{a} -function (see Theorem 6.3). At last, Appendix A and B contain some computations in Section 5.

2 Preliminaries

2.1 Let (W, S) be a Coxeter group, where S is a finite set of generators of W of order 2 satisfying braid relations. Let Γ be an abelian group (written additively) equipped with an total order $<$ such that $\gamma + \lambda_1 < \gamma + \gamma_2$ for any $\gamma \in \Gamma$ whenever $\gamma_1 < \gamma_2$. Let $L : W \rightarrow \Gamma$ be a positive weight function on W , i.e. L is a map from W to Γ such that $L(ww') = L(w) + L(w')$ if $l(ww') = l(w) + l(w')$ and $L(w) > 0$ if w is not the neutral element e , where l is the length function of Coxeter group W . Let \mathcal{A} be the group algebra of Γ . Then $\mathcal{A} = \mathbb{Z}[\Gamma] = \mathbb{Z}\{q^\gamma \mid \gamma \in \Gamma\}$ where q is viewed as a symbol. We write $\Gamma_{<0} = \{\gamma \in \Gamma \mid \gamma < 0\}$ and $\mathcal{A}_{<0} = \mathbb{Z}\{q^\gamma \mid \gamma < 0\}$; similarly for the notations $\Gamma_{>0}$, $\mathcal{A}_{>0}$, $\Gamma_{\leq 0}$, $\mathcal{A}_{\leq 0}$, etc.. Define the degree map $\deg : \mathcal{A} \rightarrow \Gamma$ on \mathcal{A} as usual:

$$\deg\left(\sum_{\gamma \in \Gamma} a_\gamma q^\gamma\right) = \max\{\gamma \mid a_\gamma \neq 0\}.$$

This is well-defined since Γ is totally ordered.

2.2 The Hecke algebra \mathcal{H} associated with (W, S, L) is an \mathcal{A} -algebra with \mathcal{A} -basis $\{T_w \mid w \in W\}$ and relations

$$\begin{aligned} T_w T_{w'} &= T_{ww'} \text{ if } l(ww') = l(w) + l(w'), \\ (T_s + q^{-L(s)})(T_s - q^{L(s)}) &= 0 \text{ if } s \in S. \end{aligned}$$

There is a \mathbb{Z} -algebra involution $\bar{}$ of \mathcal{H} , called bar involution, such that $\bar{T}_w = T_{w^{-1}}^{-1}$, $\bar{q}^\gamma = q^{-\gamma}$ for $w \in W$, $\gamma \in \Gamma$. This bar involution is used to define the well known Kazhdan-Lusztig basis: there is a unique \mathcal{A} -basis of \mathcal{H} such that

$$\begin{aligned} \bar{C}_w &= C_w, \\ C_w &\equiv T_w \pmod{\mathcal{H}_{<0}}, \end{aligned}$$

where $\mathcal{H}_{<0} := \bigoplus_{w \in W} \mathcal{A}_{<0} T_w$.

Using Kazhdan-Lusztig basis, one can define partial order $<_{\mathcal{L}}$ on W , which is generated by the relation $x \stackrel{\mathcal{L}}{<} y$, where $x \stackrel{\mathcal{L}}{<} y$ if there exist $z \in W$ such that C_x appears in the product $C_z C_y$. The partial order $<_{\mathcal{L}}$ induces naturally an equivalence relation on W : $x \stackrel{\mathcal{L}}{\sim} y$ if and only if $x <_{\mathcal{L}} y$ and $y <_{\mathcal{L}} x$. The corresponding equivalence class are called left cells.

Similarly, one can define preoder $<_{\mathcal{R}}$ on W . Equivalently, $x <_{\mathcal{R}} y$ if and only if $x^{-1} <_{\mathcal{L}} y^{-1}$. The corresponding equivalence relation is denoted by $\stackrel{\mathcal{R}}{\sim}$ and the corresponding equivalence class are called right cells.

At last, the preoder $<_{\mathcal{LR}}$ is generated by $<_{\mathcal{L}}$ and $<_{\mathcal{R}}$. The corresponding equivalence relation is denoted by $\stackrel{\mathcal{LR}}{\sim}$ and the corresponding equivalence

classes are called two-sided cells. A two-sided cell is usually denoted by \mathbf{c} in this paper.

2.3 We now introduce some notations for latter use.

Notation 2.4 (i) Obviously, the preorder $\leq_{\mathcal{LR}}$ on W induces an partial order on the set of two-sided cells. We will denote this order by \leq .

(ii) For $x \in W$ and a two-sided cell \mathbf{c} , we write $x < \mathbf{c}$ if $x \notin \mathbf{c}$ and $x <_{\mathcal{LR}} w$ for any $w \in \mathbf{c}$. And we write $x \leq \mathbf{c}$ if $x <_{\mathcal{LR}} w$ for any $w \in \mathbf{c}$.

(iii) Let $\mathcal{H}_{<\mathbf{c}} = \bigoplus_{z < \mathbf{c}} \mathcal{A}C_z$. Then $\mathcal{H}_{<\mathbf{c}}$ is a two-sided ideal of \mathcal{H} . We have similar meaning for $\mathcal{H}_{\leq \mathbf{c}}$.

(iii) For $a, b \in \mathcal{H}$,

- we write $a \equiv b$ for “ $a - b \in \mathcal{H}_{<0}$ ”, or equivalently for “ $a = b \pmod{\mathcal{H}_{<0}}$ ”;
- We write $a \dot{=} b$ for “ $a - b \in \mathcal{H}_{<\mathbf{c}}$ ”, or equivalently “ $a = b \pmod{\mathcal{H}_{<\mathbf{c}}}$ ”, or “ $a = b$ in $\mathcal{H}/\mathcal{H}_{<\mathbf{c}}$ ”;
- We write $a \ddot{=} b$ for “ $a - b \in \mathcal{H}_{<0} + \mathcal{H}_{<\mathbf{c}}$ ”, or equivalently “ $a = b \pmod{\mathcal{H}_{<0} + \mathcal{H}_{<\mathbf{c}}}$ ”, where $\mathcal{H}_{<0} + \mathcal{H}_{<\mathbf{c}} = \{a + b \mid a \in \mathcal{H}_{<0}, b \in \mathcal{H}_{<\mathbf{c}}\}$.

(iv) For $\gamma \in \Gamma$, we write $\xi_\gamma = q^\gamma - q^{-\gamma}$, $\eta_\gamma = q^\gamma + q^{-\gamma}$.

The following lemma is useful for the proof and calculation.

Lemma 2.5 If $a, b \in \mathcal{H}$ are both bar invariant in $\mathcal{H}/\mathcal{H}_{<\mathbf{c}}$ and $a = b \pmod{\mathcal{H}_{<0} + \mathcal{H}_{<\mathbf{c}}}$, then $a = b \pmod{\mathcal{H}_{<\mathbf{c}}}$. Using notations in 2.4, if $a \dot{=} \bar{a}$, $b \dot{=} \bar{b}$ and $a \ddot{=} b$ then $a \dot{=} b$.

Proof. The lemma is equivalent to that if $a - b \in \mathcal{H}_{<0} + \mathcal{H}_{<\mathbf{c}}$ and is bar invariant in $\mathcal{H}/\mathcal{H}_{<\mathbf{c}}$, then $a - b \in \mathcal{H}_{<\mathbf{c}}$. Hence we only need to prove the following claim

$$(2.1) \quad \text{If } a \in \mathcal{H}_{<0} + \mathcal{H}_{<\mathbf{c}} \text{ and } \bar{a} - a \in \mathcal{H}_{<\mathbf{c}}, \text{ then } a \in \mathcal{H}_{<\mathbf{c}}.$$

By linear independence, we have unique $\alpha_y, \alpha_z \in \mathcal{A}$ such that

$$a = \sum_{y \not< \mathbf{c}} \alpha_y C_y + \sum_{z < \mathbf{c}} \alpha_z C_z.$$

Let $a' = \sum_{y \not< \mathbf{c}} \alpha_y C_y$. Since $a \in \mathcal{H}_{<0} + \mathcal{H}_{<\mathbf{c}}$, we have $\alpha_y \in \mathcal{A}_{<0}$ and hence

$$(2.2) \quad a' \in \mathcal{H}_{<0}.$$

On the other hand, since $\bar{a} - a \in \mathcal{H}_{<\mathbf{c}}$, we have $\bar{a}' - a' \in \mathcal{H}_{<\mathbf{c}} \cap (\bigoplus_{y \not< \mathbf{c}} \mathcal{A}C_y) = 0$, and hence

$$(2.3) \quad \bar{a}' = a'.$$

Combining (2.2) and (2.3), we get $a' = 0$, see [Lus03, §5(e)]. This completes the proof of the claim (2.1). \square

2.6 There is an anti-involution b of \mathcal{A} -algebras on the Hecke algebra \mathcal{H} such that $T_w^b = T_w^{-1}$. It is obvious that $(hC_w)^b = C_{w^{-1}}h^b$ for any $h \in \mathcal{H}$.

Define the \mathcal{A} -linear map $\tau : \mathcal{H} \rightarrow \mathcal{A}$ such that $\tau(T_x) = \delta_{x,e}$. It is well known that $\tau(T_x T_y) = \delta_{x,y^{-1}}$ and $\tau(hh') = \tau(h'h)$ for any $h, h' \in \mathcal{H}$. One can check that $\tau(C_x C_y) = \delta_{x,y^{-1}} \pmod{\mathcal{H}_{<0}}$.

2.7 Let $P_{y,w}, y \in W$ be the elements in $\mathcal{A}_{\leq 0}$ such that $C_w = \sum_{y \in W} P_{y,w} T_y$. Define $\Delta(z)$ to be an element of $\Gamma_{\geq 0}$ and $n_z \in \mathbb{Z}$ such that $P_{e,z}$ (e is the neutral element of W) is of the form

$$P_{e,z} = n_z q^{-\Delta(z)} + \text{lower degree terms.}$$

Let $h_{x,y,z} \in \mathcal{A}$ be defined by $C_x C_y = \sum_z h_{x,y,z} C_z$. Then the \mathbf{a} -function $\mathbf{a} : W \rightarrow \Gamma \cup \{\infty\}$ on W is defined by

$$\mathbf{a}(z) := \sup\{\deg(h_{x,y,z}) \mid x, y \in W\}.$$

If $\mathbf{a}(z)$ is finite, $h_{x,y,z}$ can be written as the form

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} q^{\mathbf{a}(z)} + \text{lower degree terms, } \gamma_{x,y,z^{-1}} \in \mathbb{Z}$$

Recall that the \mathbf{a} -function is always bounded for affine Weyl groups (with any parameters), see [Lus85].

Define $\mathcal{D} := \{z \in W \mid \mathbf{a}(z) = \Delta(z)\}$.

Now we recall Lusztig's conjectures P1-P15 ([Lus03, §14.2]). The following is actually a reformulation of P1-P15.

2.8 Conjectures $(P1)_{\leq \mathbf{c}} - (P15)_{\leq \mathbf{c}}$. Let (W, S, L) be a Coxeter group with positive weight function and with bounded \mathbf{a} -function. Fix a two-sided cell \mathbf{c} . Then

$(P1)_{\leq \mathbf{c}}$. For any $z \leq \mathbf{c}$, we have $\mathbf{a}(z) \leq \Delta(z)$.

$(P2)_{\leq \mathbf{c}}$. If $q \in \mathcal{D}_{\leq \mathbf{c}} = \mathcal{D} \cap \{x \mid x \leq \mathbf{c}\}$, and $x, y \leq \mathbf{c}$ are such that $\gamma_{x,y,q} \neq 0$, then $x = y^{-1}$.

$(P3)_{\leq \mathbf{c}}$. For any $y \leq \mathbf{c}$, there exists uniquely $q \in \mathcal{D}_{\leq \mathbf{c}}$ such that $\gamma_{y^{-1},y,q} \neq 0$.

$(P4)_{\leq \mathbf{c}}$. If $z, z' \leq \mathbf{c}$ and $z' \leq_{LR} z$, then $\mathbf{a}(z') \geq \mathbf{a}(z)$. In particular if $z' \sim_{LR} z$ then $\mathbf{a}(z) = \mathbf{a}(z')$.

$(P5)_{\leq \mathbf{c}}$. If $q \in \mathcal{D}_{\leq \mathbf{c}}, y \leq \mathbf{c}$, $\gamma_{y^{-1},y,q} \neq 0$, then $\gamma_{y^{-1},y,q} = n_q = \pm 1$.

$(P6)_{\leq \mathbf{c}}$. For any $q \in \mathcal{D}_{\leq \mathbf{c}}$, we have $q^2 = 1$.

$(P7)_{\leq \mathbf{c}}$. For any $x, y, z \leq \mathbf{c}$, we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.

$(P8)_{\leq \mathbf{c}}$. For any $x, y, z \leq \mathbf{c}$, $\gamma_{x,y,z} \neq 0$ implies that $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z^{-1}$, $z \sim_{\mathcal{L}} x^{-1}$.

$(P9)_{\leq \mathbf{c}}$. For any $z, z' \leq \mathbf{c}$, if $z' \leq_{\mathcal{L}} z$, $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{L}} z$.

$(P10)_{\leq \mathbf{c}}$. For any $z, z' \leq \mathbf{c}$, if $z' \leq_{\mathcal{R}} z$, $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{R}} z$.

$(P11)_{\leq \mathbf{c}}$. For any $z, z' \leq \mathbf{c}$, if $z' \leq_{\mathcal{LR}} z$, $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{LR}} z$.

$(P12)_{\leq \mathbf{c}}$. Let $I \subseteq S$. If $y \in W_I$ and $y \leq \mathbf{c}$, then the \mathbf{a} -value of y in W_I is the same as that in W .

$(P13)_{\leq \mathbf{c}}$. Any left cell $\Theta \subseteq \{x \mid x \leq \mathbf{c}\}$ contains a unique element q in $\mathcal{D}_{\leq \mathbf{c}}$. And for each $y \in \Theta$ we have $\gamma_{y^{-1},y,q} \neq 0$. Similar property holds for right cells.

$(P14)_{\leq \mathbf{c}}$. For any $z \leq \mathbf{c}$, we have $z \sim_{\mathcal{LR}} z^{-1}$.

$(P15)_{\leq \mathbf{c}}$. For any $w, w' \in W$ and any x, y such that $x \sim_{\mathcal{LR}} y \leq \mathbf{c}$, we have

$$\sum_{z \in W} h_{x,w',z} \otimes h_{w,z,y} = \sum_{z \in W} h_{z,w',y} \otimes h_{w,x,z} \in \mathcal{A} \otimes_{\mathbb{Z}} \mathcal{A}.$$

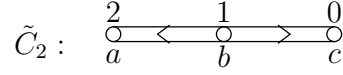


Figure 1: Dynkin diagram of \tilde{C}_2 .

We will need a weak version of $(P8)_{\leq \mathbf{c}}$:

$(P8')_{\leq \mathbf{c}}$. For any $x, y, z \leq \mathbf{c}$, $\gamma_{x,y,z} \neq 0$ implies that $y \sim_{\mathcal{L}} z^{-1}$.

If \mathbf{c} is the highest two-sided cell $\{e\}$, we simply denote $(P1)_{\leq \mathbf{c}} - (P15)_{\leq \mathbf{c}}$ by P1-P15, respectively.

3 Partition into cells

In this section we will summarize the partition of two-sided cells of affine Weyl group of type \tilde{B}_2 , and \tilde{G}_2 which is due to Guilhot, see [Gui10, Gui08]. Then we conclude some easy properties of these cells for latter use.

3.1 Assume first that W is an affine Weyl group of type \tilde{C}_2 with the set of simple reflections $S = \{s_0, s_1, s_2\}$ and the relations $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$, $s_2 s_1 s_2 s_1 = s_1 s_2 s_1 s_2$ and $s_0 s_2 = s_2 s_0$. The weight function $L : W \rightarrow \Gamma$ is given by $L(s_2) = a$, $L(s_1) = b$, $L(s_0) = c$ with a, b, c all in $\Gamma_{>0}$. See Figure 1.

We often use a sequence of 0,1,2 to denote an element of W . For example, if $w = s_0 s_1 s_0 s_1 s_2$, we abbreviates w by 01012, T_w by T_{01012} , and C_w by C_{01012} . The empty sequence denotes the neutral element e of W .

If $A, B \subseteq W$, we write AB for the set $\{xy \mid x \in A, y \in B\}$; similar for ABC with $A, B, C \subseteq W$. If we write AB , we automatically have $l(xy) = l(x) + l(y)$ for any $x \in A, y \in B$ unless otherwise specified.

We denote by \leq_D the Duflo order on W . In other words,

$$x \leq_D y \text{ if and only if } l(x^{-1}y) = l(y) - l(x).$$

If $p \in W$, we write

$$U(p) = \{w \in W \mid w \leq_D p^k \text{ for some } k \in \mathbb{N}\}.$$

3.2 We classify the two-sided cells of the affine Weyl group of type \tilde{C}_2 for various positive parameters ($a, b, c > 0$ and $a \geq c$) into the following types([Gui08]).

- (1) Let $\mathbf{c} = BdU$, $U = U(p)$ with B, d, p taking elements in the following table.

cases	parameter condition	B	d	p
i	$a - c > 2b$	$\{e, 1, 01, 101\}$	02	1012
ii	$0 < a - c < 2b$	$\{e, 2, 12, , 012\}$	1010	2101
iii	$ a - c < b, a + c > b$	$\{e, 1, 01, , 21\}$	02	102
iv	$a + c < b$	$\{e, 0, 2, 02\}$	1	021
v	$a - c > b$	$\{e, 0, 10, 010\}$	212	012
vi	$a > c, a + c > 2b$	$\{e, 1, 01, 101\}$	2	1012
vii	$a > c, a + c < 2b$	$\{e, 2, 12, , 012\}$	101	2101

Then \mathbf{c} is a two-sided cell. Set $D = \{d\}$.

- (2) Let $\mathbf{c} = BdB^{-1}$ with B, d taking elements in the following tables.

cases	condition	B	d
i	$a < b, c < b, a + c > b$	$\{e, 0, 2\}$	1
ii	$c < a < b$	$\{e, 0\}$	121
iii	$a > b > c$	$\{e, 0\}$	1
iv	$b < a - c < 2b$	$\{e, 1, 01\}$	02
v	$a > b, a + c < 2b$	$\{e, 1, 01\}$	2
vi	$a > c > b$	$\{e, 1\}$	0

Then \mathbf{c} is a two-sided cell. Set $D = \{d\}$ and $U = B^{-1}$.

- (3) Let $\mathbf{c} = B_1d_1U_1 \cup B_2d_2U_2$, $U_1 = U(p_1)$, $U_2 = U(p_2)$ with B_1, d_1, p_1 and B_2, d_2, p_2 taking elements in the following table.

cases	conditions	B_1	d_1	p_1	B_2	d_2	p_2
i	$a - c = 2b$	$\{e, 1, 01\}$	02	1012	$\{e\}$	1010	2101
ii	$a + c = b$	$\{e, 0, 2\}$	1	021	$\{e\}$	02	102
iii	$a - c = b$	$\{e, 1, 01\}$	02	102	$\{e\}$	212	012
iv	$a > c, a + c = 2b$	$\{e, 1, 01\}$	2	1012	$\{e\}$	101	2101
v	$a = c, a > b$	$\{e, 1\}$	2	1012	$\{e, 1\}$	0	1210
vi	$a = c, a < b$	$\{e, 2\}$	101	2101	$\{e, 0\}$	121	0121

Then \mathbf{c} is a two-sided cell. Set $D = \{d_1, d_2\}$.

- (4) Exotic cases:

- (i) If $a = b = c$, set $B_1 = \{e\}$, $d_1 = 1$, $U_1 = U(0121) \cup U(2101)$, set $B_2 = \{e\}$, $d_2 = 2$, $U_2 = U(1012) \cup \{12\}$, and set $B_3 = \{e\}$, $d_3 = 0$, $U_3 = U(1210) \cup \{10\}$. Then $\mathbf{c} = B_1d_1U_1 \cup B_2d_2U_2 \cup B_3d_3U_3$ is a two-sided cell. Set $D = \{0, 1, 2\}$.
- (ii) If $a > b = c$, then

$$\mathbf{c} = \{1, 10, 101\} \cup \{0, 01, 010\}$$

is a two-sided cell. Set $B_1 = \{e\}$, $d_1 = 1$, $U_1 = \{e, 0, 01\}$ and set $B_2 = \{e\}$, $d_2 = 0$, $U_2 = \{e, 1, 10\}$. Then $\mathbf{c} = B_1d_1U_1 \cup B_2d_2U_2$. Set $D = \{d_1, d_2\}$.

- (iii) If $a = b > c$, then

$$\mathbf{c} = \{1, 10, 12, 121, 1210\} \cup \{01, 010, 012, 0121, 01210\} \cup \{2, 21, 210, 212\}$$

is a two-sided cell. Set $B_1 = \{e, 0\}$, $d_1 = 1$, $U_1 = \{0, 2, 21, 210\}$, and set $B_2 = \{e\}$, $d_2 = 2$, $U_2 = \{e, 1, 10, 12\}$. Then $\mathbf{c} = B_1d_1U_1 \cup B_2d_2U_2$. Set $D = \{d_1, d_2\}$.

- (5) All the cells with one element are listed as follows:

- $\{1\}$ when $a, c > b$;
- $\{010\}$ when $a, c > b$;
- $\{1010\}$ when $a - c > 2b$;
- $\{212\}$ when $b, c < a$ and $a < b + c$;
- $\{0\}$ when $c < b$ and $a \geq c$;
- $\{101\}$ when $a + c > 2b$ and $c < b$;
- $\{2\}$ when $a \geq c$ and $a < b$;

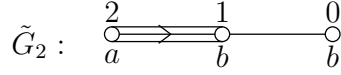


Figure 2: Dynkin diagram of \tilde{G}_2 .

– $\{02\}$ when $a + c < b$.

(6) The lowest two-sided cell \mathbf{c} :

– if $a > c$, then

$$\mathbf{c} = BdU$$

with $B = \{b \mid b^{-1} \leq_D 010210\}$, $d = 1212$, and $U = \{u \in W \mid l(1212u) = l(u) + l(1212)\}$;

– if $a = c$, then

$$\mathbf{c} = B_1 d_1 U_1 \cup B_2 d_2 U_2$$

with $d_1 = 1212$, $d_2 = 1010$, $U_1 = \{e, 0, 10, 210\}$, $B_2 = \{e, 2, 12, 012\}$, $U_1 = \{u \in W \mid l(1212u) = l(u) + l(1212)\}$, and $U_2 = \{u \in W \mid l(1010u) = l(u) + l(1010)\}$.

These are all possible two-sided cells of \tilde{C}_2 .

3.3 Assume now that W is the affine Weyl group of type \tilde{G}_2 with generators s_0, s_1, s_2 and relations $s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1$, $s_1 s_0 s_1 = s_0 s_1 s_0$, and $s_0 s_2 = s_2 s_0$. The weights are $L(s_2) = a$, $L(s_1) = b = L(s_0)$ with $a, b \in \Gamma_{>0}$. See Figure 2.

We also abbreviate an element of W using the sequences of 0,1,2 associated with its reduced expressions. We now describe the two-sided cells of \tilde{G}_2 .

(1) Let $\mathbf{c} = BdU$, $U = U(p)$ with B, d, p taking elements in the following table.

cases	condition	B	d	p
i	$2a > 3b$	$\{e, 0, 10, 210, 1210, 01210\}$	21212	01212
ii	$2a < 3b$	$\{e, 1, 21, 121, 0121, 2121\}$	02	12102
iii	$a > 2b$	$\{e, 1, 01, 21, 121, 0121\}$	02	102
iv	$a < 2b$	$\{e, 2, 12, 212, 1212, 01212\}$	101	210
v	$2a > 3b, a < 2b$	$\{e, 1, 21, 121, 0121\}$	02	e
vi	$a < b$	$\{e, 0\}$	12121	e

Then \mathbf{c} is a two-sided cell. Set $D = \{d\}$.

(2) Let $\mathbf{c} = B_1 d_1 U_1 \cup B_2 d_2 U_2$, $U_1 = U(p_1)$, $U_2 = U(p_2)$ with B_1, d_1, p_1 and B_2, d_2, p_2 taking values in the following table.

cases	condition	B_1	d_1	p_1	B_2	d_2	p_2
i	$2a = 3b$	$\{e, 1, 21, 121, 0121\}$	02	12102	$\{e\}$	21212	01212
ii	$a = 2b$	$\{e, 1, 21, 121, 0121\}$	02	102	$\{e\}$	101	210

Then \mathbf{c} is a two-sided cell. Set $D = \{d_1, d_2\}$.

(3) Exotic cases.

(i) If $a > b$, then

$$\mathbf{c} = \{1, 10\} \cup \{0, 01\}$$

is a two-sided cell; $D = \{1, 0\}$. Set $B_1 = B_2 = \{e\}$, $d_1 = 1$, $d_2 = 0$, $U_1 = \{e, 0\}$, $U_2 = \{e, 1\}$. Then $\mathbf{c} = B_1 d_1 U_1 \cup B_2 d_2 U_2$.

(ii) If $a < b$, then

$$\mathbf{c} = \{21, 210, 212, 2121, 21210, 21212\} \cup \{1, 12, 121, 1212, 10, 1210\} \\ \cup \{0, 01, 012, 0121, 01210, 01212\}$$

is a two-sided cell; $D = \{1, 0\}$. Set $B_1 = \{e, 2\}$, $d_1 = 1$, $U_1 = \{e, 2, 0, 21, 212, 210\}$ and set $B_2 = \{e\}$, $d_2 = 0$, $U_2 = \{e, 1, 12, 121, 1210, 1212\}$. Then $\mathbf{c} = B_1 d_1 U_1 \cup B_2 d_2 U_2$.

(iii) If $a = b$, then

$$\mathbf{c} = \{1, 10, 12, 121, 1210, 1212, 12121, 121210\} \\ \cup \{0, 01, 012, 0121, 01210, 01212, 012121, 0121210\} \\ \cup \{2, 21, 210, 212, 2121, 21210, 21212\}$$

is a two-sided cell; $D = \{1, 0, 2\}$. Then set $d_1 = 1$, $d_2 = 0$, $d_3 = 2$. Then $\mathbf{c} = B_1 d_1 U_1 \cup B_2 d_2 U_2 \cup B_3 d_3 U_3$ where $B_1 = B_2 = B_3 = \{e\}$.

(iv) If $a > b$, then

$$\mathbf{c} = \{e, 1, 01\}\{2\}\{e, 12\}\{e, 1, 10\}.$$

is a two-sided cell. Set $d = 2$, $B = \{e, 1, 01\}$, $U = \{e, 1, 10, 12, 121, 1210\}$, $D = \{2\}$. Then $\mathbf{c} = BdU$.

(4) The two-sided cells with one element:

- $\{2\}$ when $a < b$;
- $\{21212\}$ when $2b < 2a < 3b$;
- $\{101\}$ when $a > 2b$.

(5) The lowest two-sided cell:

$$\mathbf{c} = BdU$$

where $B = \{b \mid b^{-1} \leq_D 0121201210\}$, $d = 121212$, $U = \{u \in W \mid l(121212u) = l(121212) + l(u)\}$.

These are all possible two-sided cells of \tilde{G}_2 .

3.4 We conclude this section with following observations.

Lemma 3.5 *Let \mathbf{c} be a two-sided cell of an affine Weyl group W of type \tilde{C}_2 or \tilde{G}_2 with positive weight function. Then*

(i) \mathbf{c} can be written in the form

$$\mathbf{c} = \bigcup_{d \in D} B_d d U_d$$

where D , B_d , U_d ($d \in D$) are subsets of W which have been listed in **3.2**, **3.3** case by case. And this form satisfies

- For each $d \in D$, $b \in B_d$ and $u \in U_d$, we have $l(bdu) = l(b) + l(d) + l(u)$.
- $e \in B_d$, $e \in U_d$, and $B_d^{-1} \subseteq U_d$.
- Each $d \in D$ is an involution element of some finite parabolic subgroups of W such that $l(su) = l(u) + 1$ for any $u \in U_d$, and $s \in S$ with $s \leq d$.

- (ii) If $w \notin U_d$ satisfying $l(sw) = l(w) + 1$ for $s \in S$ with $s \leq d$, then $dw \notin \mathbf{c}$.
- (iii) The right cells in \mathbf{c} are all of the form

$$\{bdu \mid u \in U_d\}, d \in D, b \in B_d.$$

Proof. (i) is just a direct observation from subsection **3.2, 3.3**. (ii) follows from [Gui10, Rem,6.7]. (iii) is one of the conclusion in [Gui10]. \square

4 Decomposition formula

In this section, we need the following assumptions on a two-sided cell \mathbf{c} of a Coxeter group W .

Assumption 4.1 *Let \mathbf{c} be a two-sided cell of W . We assume that there exist subsets D, B_d, U_d ($d \in D$) of \mathbf{c} such that*

- (i) (a) $\mathbf{c}^{-1} = \mathbf{c} = \bigsqcup_{d \in D} B_d d U_d$.
- (b) For each $d \in D, b \in B_d$ and $u \in U_d$, we have $l(bdu) = l(b) + l(d) + l(u)$.
- (c) The neutral element $e \in B_d, U_d$, and $B_d^{-1} \subseteq U_d$.
- (d) If $w \notin U_d$ satisfying $l(sw) = l(w) + 1$ for any simple reflection $s \leq d$, then $dw \notin \mathbf{c}$.
- (ii) Each $d \in D$ is an involution element of some finite parabolic subgroups of W with the following properties:
 - (a) $l(su) = l(u) + 1$ for any $s \leq d, u \in U_d$.
 - (b) For any $s \leq d$, we have $T_s C_d \in \mathcal{A}C_d + \mathcal{H}_{<\mathbf{c}}$.
 - (c) $h_{d,d,d} \neq 0$.
- (iii) For any $d \in D$, the set dU_d is a right cell of W .
- (iv) For any $d \in D, b \in B_d, u \in U_d$, we have

$$T_b C_d T_u = T_{bdu} \mod \mathcal{H}_{<0} + \mathcal{H}_{<\mathbf{c}}.$$

It follows from Lemma **3.5** that Assumption **4.1** has been hold for affine Weyl groups of rank 2, except Assumption **4.1** (ii,b-c) and (iv). We will verify them with some calculations in the next section.

The main result of this section can roughly be stated as follows. Under the assumption **4.1**, we have product decomposition:

$$h_{d,d,d} C_{bdu} = C_{bd} C_{du} \mod \mathcal{H}_{<\mathbf{c}}$$

for any $d \in D, b \in B_d, u \in U_d$.

Lemma 4.2 *Keep Assumption 4.1.*

- (i) For any $d \in D, u \in U_d$, we have $C_d T_u = T_{du} \mod \mathcal{H}_{<0}$.
- (ii) For any $d \in D$, we have $C_d C_d = h_{d,d,d} C_d \mod \mathcal{H}_{<\mathbf{c}}$.
- (iii) The set $\{C_d T_u \mid u \in U_d\}$ is a \mathcal{A} -linearly independent subset of $\mathcal{H}/\mathcal{H}_{<\mathbf{c}}$.

Proof. The statement (i) follows directly from Assumption 4.1 (ii,a). The statement (ii) follows directly from Assumption 4.1 (ii,b). To prove (iii), we only need to prove that

$$\{C_d T_u \mid u \in U_d\} \cup \{C_z \mid z < \mathbf{c}\} \text{ is linearly independent in } \mathcal{H},$$

which follows from the fact that

$$C_d T_u \in T_{du} + \sum_{y < du} \mathcal{A} T_y, \text{ and } C_z \in T_z + \sum_{y < z} \mathcal{A} T_y.$$

□

Definition 4.3 Fix an element $d \in D$. For any $w \in W$, we say $y <_U w$ if and only if $y < w$ in Bruhat order and $y \in U_d$. And we write $y \leq_U w$ if $y <_U w$ or $y = w$. In other words, for any $w \in W$, we have $\{y \mid y \leq_U w\} = \{y \in U_d \mid y < w\} \cup \{w\}$.

Lemma 4.4 Fix a two-sided cell \mathbf{c} as the Assumption 4.1 and an element $d \in D$. Let $w \in W$ satisfy the condition that for any element $s \in S$ such that $s \leq d$ we have $l(sw) = 1 + l(w)$ (the elements of U_d satisfy this condition). Then

(i) There exists $r_{y,w} \in \mathcal{A}$ such that

$$\overline{C_d T_w} = C_d T_w + \sum_{y <_U w} r_{y,w} C_d T_y \pmod{\mathcal{H}_{<\mathbf{c}}}.$$

(ii) There exists a unique element $F_w \in \mathcal{H}$ such that

$$(a) \ F_w = T_w + \sum_{y <_U w} p_{y,w} T_y \text{ with } p_{y,w} \in \mathcal{A}_{<0}.$$

$$(b) \ C_d F_w = C_d T_w \pmod{\mathcal{H}_{<\mathbf{c}}}.$$

(iii) If $w \notin U_d$, then $C_d T_w \in \sum_{y <_U w} \mathcal{A} C_d T_y + \mathcal{H}_{<\mathbf{c}}$.

Proof. We use induction on the length $n = l(w)$ of w . The proof forms an interesting loop. The lemma for $n = 0$ is obvious. We will use the notation $(iii)_{<n}$ to express that the statement (iii) holds for all w with length $< n$. We will prove that $(iii)_{<n}$ implies $(i)_n$; $(i)_{\leq n}$ implies $(ii)_n$; and $(ii)_n$ implies $(iii)_n$. Then we can conclude the lemma.

$(iii)_{<n} \implies (i)_n$. It is well-known that $\overline{T_w} = T_w + \sum_{y' < w} R_{y',w} T_{y'}$ with $R_{y',w} \in \mathcal{A}$. By the Assumption 4.1 (ii,b), we have

$$\overline{C_d T_w} \in C_d T_w + \sum_y \mathcal{A} C_d T_y + \mathcal{H}_{<\mathbf{c}},$$

where y takes over the elements such that

$$y < w, \text{ and } l(sy) = 1 + l(y) \text{ for any simple reflection } s \leq d.$$

Applying $(iii)_{<n-1}$ to y , we immediately get $(i)_n$.

$(i)_{\leq n} \implies (ii)_n$. The basic idea follows from the construction of Kazhdan-Lusztig basis. We first prove that

(*) there exists a unique element $F_u \in \mathcal{H}$ such that (a) holds and $C_d F_u$ is invariant in $\mathcal{H}/\mathcal{H}_{<\mathbf{c}}$ under the bar involution $\bar{}$.

Let w' be such that $l(w') \leq n$ and such that $l(sw') = l(w') + 1$ for any $s \leq d$ with $s \in S$. By (i) _{$\leq n$} , we have

$$\overline{C_d T_{w'}} = \sum_{y \leq_U w'} r_{y,w'} C_d T_y \pmod{\mathcal{H}/\mathcal{H}_{<\mathbf{c}}},$$

where we convention that $r_{w',w'} = 1$. Then

$$C_d T_{w'} \doteq \sum_{y \leq_U w'} \bar{r}_{y,w'} \overline{C_d T_y} \doteq \sum_{y' \leq_U w'} \left(\sum_{y: y' \leq_U y \leq_U w'} r_{y',y} \bar{r}_{y,w'} \right) C_d T_{y'}.$$

Recall from Notation **2.4** that “ \doteq ” means “equals by modulo $\mathcal{H}_{<\mathbf{c}}$ ”. By linear independence (see Lemma **4.2**(iii)), we obtain

$$(4.1) \quad \sum_{y: y' \leq_U y \leq_U w'} r_{w',y} \bar{r}_{y,w'} = \delta_{y',w'}.$$

(Note that when $y' = w'$, the equation holds by our convention $r_{w',w'} = 1$.)

Write $F_w = \sum_{y \leq_U w} p_{y,w} T_y$ and convention that $p_{w,w} = 1$. We prove now claim (*) by solving equations. It is easy to see that

$$\overline{C_d F_w} \doteq \sum_{y' \leq_U w} \left(\sum_{y: y' \leq_U y \leq_U w} r_{y',y} \bar{p}_{y,w} \right) C_d T_{y'}.$$

By Lemma **4.2**(iii), we see that to require $C_d F_u$ is invariant in $\mathcal{H}/\mathcal{H}_{<\mathbf{c}}$ under bar involution is equivalent to require that

$$\sum_{y: y' \leq_U y \leq_U w} r_{y',y} \bar{p}_{y,w} = p_{y',w}, \forall y' \leq_U w$$

and this is further equivalent to

$$(4.2) \quad p_{y',w} - \bar{p}_{y',w} = \sum_{y: y' <_U y \leq_U w} r_{y',y} \bar{p}_{y,w}.$$

We solve $p_{y',w}$ by induction on $l(w) - l(y')$. If $l(w) - l(y') = 0$ then (4.2) agrees with our convention $p_{w,w} = 1$. Assume that we have known $p_{y,w}$ for all y satisfying $y' <_U y \leq_U w$. Then $p_{y,w} = \sum_{v: y \leq_U v \leq_U w} r_{y,v} \bar{p}_{v,w}$. So

$$\begin{aligned} \overline{\sum_{y: y' <_U y \leq_U w} r_{y',y} \bar{p}_{y,w}} &= \sum_{y: y' <_U y \leq_U w} \bar{r}_{y',y} \left(\sum_{v: y \leq_U v \leq_U w} r_{y,v} \bar{p}_{v,w} \right) \\ &= \sum_{v: y' <_U v \leq_U w} \left(\sum_{y: y' <_U y \leq_U v} \bar{r}_{y',y} r_{y,v} \right) \bar{p}_{v,w} \\ &= \sum_{v: y' <_U v \leq_U w} (-r_{y',v}) \bar{p}_{v,w} \text{ (by (4.1))} \\ &= - \sum_{y: y' <_U y \leq_U w} r_{y',y} \bar{p}_{y,w} \end{aligned}$$

In other words, the right hand side of (4.2) is anti-invariant under bar involution. Then we can see that $p_{y,w}$ is just the part of the negative degrees of the Laurent polynomial $\sum_{y:y' <_U y \leq_U w} r_{y',y} \bar{p}_{y,w}$. So $p_{y',w}$ uniquely determined by (4.2) and hence (*) holds.

By Lemma 4.2(i) and the assumption on w , we have

$$C_d F_w = T_{dw} = C_{dw} \mod \mathcal{H}_{<0}.$$

Since $C_d F_w$ is invariant in $\mathcal{H}/\mathcal{H}_{<\mathbf{c}}$ under the bar involution, we have $C_d F_w = C_{dw} \mod \mathcal{H}_{<\mathbf{c}}$, see Lemma 2.5. This proves (ii)_n.

(ii)_n \implies (iii)_n. Let $w \notin U_d$ but $l(sw) = l(w) + 1$ for any $s \in S$ with $s \leq d$. By the Assumption 4.1(i,d), we have $dw \notin \mathbf{c}$. Then by (ii)_n we have $C_d F_w = C_{dw} = 0 \mod \mathcal{H}_{<0}$. Hence $C_d T_w = -\sum_{y <_U w} p_{y,w} C_d T_y \mod \mathcal{H}_{<0}$. Then (iii)_n follows. This completes a loop of the induction. \square

Corollary 4.5 *Let $d \in D$, $u, v^{-1} \in U_d$. Then*

$$C_{du} = C_d F_u \mod \mathcal{H}_{<\mathbf{c}}$$

$$C_{vd} = E_v C_d \mod \mathcal{H}_{<\mathbf{c}}$$

where $E_v = (F_{v^{-1}})^{\flat}$, see 2.6 for the definition of the map $^{\flat}$.

Proof. The first equation is just a weak version of Lemma 4.4(ii). Note that w in Lemma 4.4 is not necessarily in U_d . The stronger version there are required in the proof.

For second equation, one only needs to apply the anti-involution $^{\flat}$ to the first. \square

Corollary 4.6 *Let $d \in D$, $u, v^{-1} \in U_d$.*

(i) *There exist unique $q_{u',u} \in \mathcal{A}_{\leq 0}$ with $u' \leq_U u$ such that*

$$C_d T_u = \sum_{u' \leq_U u} q_{u',u} C_d F_{u'} \mod \mathcal{H}_{<\mathbf{c}}$$

and $q_{u,u} = 1$, $q_{u',u} \in \mathcal{A}_{<0}$ for $u' \neq u$.

(ii)

$$T_v C_d = \sum_{v'^{-1} \leq_U v^{-1}} q_{v'^{-1}, v^{-1}} E_{v'} C_d \mod \mathcal{H}_{<\mathbf{c}}.$$

Proof. By Lemma 4.4(ii), we have

$$C_d T_u = C_d F_u - \sum_{u' <_U u} p_{u',u} C_d T_{u'} \mod \mathcal{H}_{<\mathbf{c}}$$

Then (i) follows immediately by induction. We get (ii) by applying the anti-involution $^{\flat}$ to (i). \square

Theorem 4.7 (Decomposition formula) *Keep Assumption 4.1. Let $d \in D$, $b \in B_d$, $u \in U_d$. We have the decomposition*

$$C_{bdu} = E_b C_d F_u \mod \mathcal{H}_{<\mathbf{c}},$$

where E_b, F_u is given by Corollary 4.5. Equivalently, we have

$$C_{bd} C_{du} = h_{d,d,d} C_{bdu} \mod \mathcal{H}_{<\mathbf{c}}.$$

Proof. The proof uses induction on $l(b) + l(u)$. By Assumption 4.1(iv) and Corollary 4.6, we have

$$\begin{aligned} C_{bdu} &\equiv T_{bdu} \\ &\doteq T_b C_d T_u \\ &\doteq \sum_{b'^{-1} \leq_U b^{-1}, u' \leq_U u} q_{b',b} q_{u',u} E_{b'} C_d F_{u'}. \end{aligned}$$

(Recall the notation from 2.4(iii).) By induction hypothesis, when $b' \neq b$ or $u' \neq u$, we have $E_{b'} C_d F_{u'} \doteq C_{b'du'}$ and $(q_{b',b} q_{u',u}) \in \mathcal{A}_{<0}$ (see Corollary 4.6), and hence $q_{b',b} q_{u',u} E_{b'} C_d F_{u'} \doteq 0$. So

$$C_{bdu} \doteq \sum_{b'^{-1} \leq_U b^{-1}, u' \leq_U u} q_{b',b} q_{u',u} E_{b'} C_d F_{u'} \doteq E_b C_d F_u.$$

Now by Lemma 4.5 and Lemma 4.2(ii) we have

$$C_{bd} C_{du} \doteq (E_b C_d)(C_d F_u) \doteq h_{d,d,d} E_b C_d F_u.$$

Then one can see that $E_b C_d F_u$ is invariant in $\mathcal{H}/\mathcal{H}_{<\mathbf{c}}$ under bar involution. Therefore by Lemma 2.5 we have

$$C_{bdu} \doteq E_b C_d F_u.$$

This completes the proof. \square

Corollary 4.8 *Keep Assumption 4.1.*

- (i) *For any $d \in D$, $b \in U_d$, $\Phi_{b,d} = \{bdu \mid u \in U_d\}$ is a right cell contained in \mathbf{c} .*
- (ii) *The left cells contained in \mathbf{c} are all of the form $\Theta_{b,d} = \Phi_{b,d}^{-1}$.*
- (iii) *If $xy, x \in \mathbf{c}$ and $l(xy) = l(x) + l(y)$, then xy, x are in the same right cells. Similarly, if $xy, y \in \mathbf{c}$ and $l(xy) = l(x) + l(y)$, then xy, y are in the same left cells.*
- (iv) *For each $w \in \mathbf{c}$, there exist unique $d_1, d_2 \in D$ and $b_1 \in B_{d_1}$, $b_2 \in B_{d_2}$ such that $w \in \Phi_{b_1,d_1} \cap \Theta_{b_2,d_2}$.*
- (v) *Keep the notation in (iv). There exists unique $p_w \in \Phi_{e,d_1} \cap \Theta_{e,d_2}$ such that $w = b_1 p_w b_2^{-1}$ and $l(w) = l(b_1) + l(p_w) + l(b_2)$. In other words, we have a union without intersection*

$$\mathbf{c} = \bigsqcup_{d_1, d_2 \in D} B_{d_1} (\Phi_{e,d_1} \cap \Theta_{e,d_2}) B_{d_2}^{-1}.$$

(vi) Keep the notation in (v). We have decomposition

$$C_w = E_{b_1} C_{p_w} F_{b_2^{-1}} \mod \mathcal{H}_{<c}.$$

Proof. By Assumption 4.1(iii), $\Phi_{e,d}$ is a right cell. By decomposition formula, for any $b \in B_d, u \in U_u$, we have $C_{bdu} = E_b C_{du}$. Using 4.1(i,d), we see immediately that $\Phi_{b,d}$ is a right cell for any $b \in B_d$. Then (i) and (ii) are proved.

(iii). Let $x = bdu$. Then $xy = bd(uy)$ with $l(bduy) = l(b) + l(d) + l(uy)$, and hence $uy \in U_d$. By (i), we see that x, xy are in the same right cell.

(iv). It is obvious that Φ_{b_1,d_1} (resp. Θ_{b_2,d_2}) is the right (resp. left) cell containing w .

(v). Assume that $w \in \Phi_{b_1,d_1} \cap \Theta_{b_2,d_2}$. Let $w = b_1 d_1 u_1$. By (iii), the elements $d_1 u_1$ and w are in the same left cell, i.e. $d_1 u_1 \in \Theta_{b_2,d_2}$. Write $d_1 u_1 = u_2^{-1} d_2 b_2^{-1}$ for some $u_2 \in U_{d_2}$. By (iii) again, the elements $u_2^{-1} d_2$ and $d_1 u_1$ are in the same right cell Φ_{e,d_1} . Then the element $p_w =: u_2^{-1} d_2$ satisfies the conclusion in (v).

(vi). Write $w = b_1 p_w b_2^{-1}$ as (v). By the decomposition formula, we see that

$$C_w \doteq E_{b_1} C_{p_w b_2^{-1}} \doteq E_{b_1} C_{p_w} F_{b_2^{-1}}.$$

□

5 Prove Assumption 4.1 for affine Weyl groups of rank 2

5.1 In this section we verify Assumption 4.1 for the affine Weyl groups of type \tilde{B}_2 and \tilde{G}_2 . Then we can see the conclusions discussed in the last section all holds for them. By Lemma 3.5, we only need to verify Assumption 4.1 (ii,b-c) and (iv).

5.2 If $d \in D$ is the longest element of some finite parabolic subgroup of W , then Assumption 4.1 (ii,b-c) will automatically hold. So we only need verify the following cases for Assumption 4.1 (ii,b-c).

5.3 The case of type \tilde{C}_2 . By 3.2, there are four cases such that $d \in D$ is not the longest element of some finite parabolic subgroup.

(i) When $c < b$ ($a \geq c$), 101 is an element in D .

Since

$$\begin{aligned} C_1 C_0 C_1 &= C_1 T_0 C_1 + q^{-c} C_1 C_1 \\ &= C_1 T_0 C_1 + q^{-c} (q^b + q^{-b}) C_1 \\ &= C_1 T_0 C_1 + (q^{-b-c} - q^{c-b}) C_1 + (q^{-c+b} + q^{c-b}) C_1 \\ &= C_1 T_0 C_1 - q^{-b} \xi_c C_1 + \eta_{b-c} C_1, \end{aligned}$$

we can see that $C_1 T_0 C_1 - q^{-b} \xi_c C_1 = C_1 C_0 C_1 - \eta_{b-c} C_1$ is bar invariant and $\equiv T_{101} \mod \mathcal{H}_{<0}$ (see 2.4(iv) for notations). Hence

$$(5.1) \quad C_{101} = C_1 T_0 C_1 - q^{-b} \xi_c C_1$$

$$= T_{101} + q^{-b}(T_{10} + T_{01}) + q^{-2b}T_0 - q^{-b}\xi_c T_1 - q^{-2b}\xi_c.$$

From (5.1), we see immediately that $C_0 C_{101} = T_{0101} \pmod{\mathcal{H}_{<0}}$. So $C_0 C_{101} = C_{1010}$. Hence $T_0 C_{101} \doteq -q^{-c} C_{101}$ (note that $C_{1010} \in \mathcal{H}_{<c}$). On the other hand, $T_1 C_{101} = q^b C_{101}$ ($C_1 C_{101} = \eta_b C_{101}$) is well-known. Then Assumption 4.1 (ii,b) holds in this case. From these calculations, we see also that

$$C_{101} C_{101} \doteq (-q^{-c}\eta_b^2 - q^{-b}\xi_c \eta_b) C_{101} \doteq -\eta_b \eta_{b-c} C_{101}.$$

- (ii) When $a = c$, $a < b$, 121 is an element in D . The calculation is very similar to (i). One only need replace 0 by 2, and c by a .
- (iii) 212 is in D if and only if $b < a$. The calculations is also similar to (i). The results of calculations are as follows:

$$\begin{aligned} C_{212} &= C_2 T_1 C_2 - q^{-a} \xi_b C_2, \\ T_2 C_{212} &= q^a C_{212} \\ T_1 C_{212} &\doteq -q^{-b} C_{212} \\ C_{212} C_{212} &\doteq -\eta_a \eta_{a-b} C_{212}. \end{aligned}$$

- (iv) When $b < c$, 010 is in D . The calculation is similar to (iii). One only need to change 2 to 0, and change a to c .

5.4 The case of type \tilde{G}_2 . By 3.3 there are two cases such that $d \in D$ is not the longest element of a parabolic longest element.

- (i) When $a > b$, 21212 is in D .

$$\begin{aligned} C_2 C_1 C_2 C_1 C_2 &= C_2 T_1 T_2 T_1 C_2 + (2q^{a-b} + q^{-a} \eta_b) C_2 T_1 C_2 \\ &\quad + (q^{a-2b} \eta_a + \eta_a \eta_b q^{-a-b}) C_2 \\ 2\eta_{a-b} C_{212} &= 2\eta_{a-b} C_2 T_1 C_2 - 2\eta_{a-b} q^{-a} \xi_b C_2 \end{aligned}$$

So we have

$$\begin{aligned} &C_2 C_1 C_2 C_1 C_2 - 2\eta_{a-b} C_{212} - (\eta_{2a-2b} + 3) C_2 \\ &= C_2 T_1 T_2 T_1 C_2 - q^{-a} \xi_b C_2 T_1 C_2 + q^{-2a} (\eta_{2b} - 1) C_2. \end{aligned}$$

Then $C_{21212} = C_2 T_1 T_2 T_1 C_2 - q^{-a} \xi_b C_2 T_1 C_2 + q^{-2a} (\eta_{2b} - 1) C_2$. This implies that $C_1 C_{21212} = T_{121212} \pmod{\mathcal{H}_{<0}}$. So $C_1 C_{21212} = C_{121212}$. And hence $T_1 C_{21212} = -q^{-b} C_{21212} \pmod{\mathcal{H}_{<c}}$. On the other hand $T_2 C_{21212} = q^b C_{21212}$ is well-known. Using these calculations we have

$$\begin{aligned} C_{21212} C_{21212} &\doteq (q^{a-2b} \eta_a^2 + q^{-a-b} \xi_b \eta_a^2 + q^{-2a} (\eta_{2b} - 1) \eta_a) C_{21212} \\ &\doteq \eta_a (\eta_{2a-2b} + 1) C_{21212}. \end{aligned}$$

- (ii) When $a < b$, 12121 is in D . The calculation is similar to (i). The results of calculations are as follows

$$\begin{aligned} C_{12121} &= C_1 T_2 T_1 T_2 C_1 - q^{-b} \xi_a C_1 T_2 C_1 + q^{-2b} (\eta_{2a} - 1) C_1. \\ T_2 C_{12121} &\doteq -q^{-a} C_{12121} \\ C_{12121} C_{12121} &\doteq \eta_b (\eta_{2b-2a} + 1) C_{12121}. \end{aligned}$$

By the above calculations, we can conclude Assumptions 4.1(ii,b-c) in present case.

5.5 In remaining of this section we prove the last assumption: 4.1(iv).

The case for lowest two-sided cell has been known, see [Xie15, Cor 3.3]. So we only deal with non-lowest two-sided cells with more than one element.

The computations for cases $(\tilde{C}_{2,1,vii})$, $(\tilde{G}_{2,1,i-iv})$, $(\tilde{G}_{2,2,i-ii})$ are more complicated than other cases. We refer cases $(\tilde{C}_{2,1,vii})$, $(\tilde{G}_{2,1,i-iv})$, $(\tilde{G}_{2,2,i-ii})$ as complicated cases, and refer the remaining cases as easy cases.

5.6 Easy cases.

In the easy cases, we actually have conclusion stronger than Assumption 4.1(iv).

Proposition 5.7 *Except the cases $(\tilde{C}_{2,1,vii})$, $(\tilde{G}_{2,1,i-iv})$, $(\tilde{G}_{2,2,i-ii})$, we have*

$$(5.2) \quad T_b C_d T_u = T_{bdu} \mod \mathcal{H}_{<0}$$

Compare with Assumption 4.1(iv).

Lemma 5.8 *Let $d' < d$ with $d \in D$, $b \in B_d$, $u \in U_d$ and $\alpha \in \mathcal{A}$. If $\alpha T_b T_{d'} T_u \in \mathcal{H}_{<0}$, then for all $b'^{-1} \leq_D b^{-1}$, $\forall u' \leq_D u$ we have $\alpha T_{b'} T_{d'} T_{u'} \in \mathcal{H}_{<0}$.*

Proof. The proof is immediately from the fact that $m_{x,y,z}$ is a polynomials in $\xi_{L(s)}$, $s \in S$ with positive integral coefficients, where $T_x T_y = \sum_z m_{x,y,z} T_z$. \square

The following lemma is obvious.

Lemma 5.9 *Assume $T_x T_y = \sum_z m_{x,y,z} T_z$. Let w be an element in W such that $T_y T_w = T_{yw}$ and $T_z T_w = T_{zw}$ for all z with $m_{x,y,z} \neq 0$. Then we have $m_{x,yw,zw} = m_{x,y,z}$.*

Corollary 5.10 *let $d' < d \in D$, $b \in B_d$, $u \in U_d$. Assume that, for any $w \in W$ such that $T_u T_w = T_{uw}$ and $uw \in U_d$, we have $T_z T_w = T_{zw}$ for all z with $m_{b,d'u,z} \neq 0$. And assume that there exists $\alpha \in \mathcal{A}$ such that $\alpha T_b T_{d'} T_u \in \mathcal{H}_{<0}$. Then for all $b'^{-1} \leq_U b^{-1}$ and $u' \in U_d$ we have $\alpha T_{b'} T_{d'} T_{u'} \in \mathcal{H}_{<0}$.*

Proof. By Lemma 5.9, we have $m_{b,d'uw,zw} = m_{b,d'u,z}$. So, for all $b'^{-1} \leq_U b^{-1}$ and $u' \in U_d$ with $u \leq_U u'$, we have $\alpha T_{b'} T_{d'} T_{u'} \in \mathcal{H}_{<0}$. Then the corollary follows from Lemma 5.8. (Here we need a fact: for any $u, v \in U_d$ there exists $u' \in U_d$ such that $u \geq_U u'$ and $v \geq_U u'$. For this fact, one can check easily case by case.) \square

Now we can do some calculations case by case and then prove Assumption 4.1(iv).

Case $(\tilde{C}_{2,1,i})$. $a - c > 2b$. $B = \{e, 1, 01, 101\}$, $d = 02$, $U = U(1012)$.

$$\begin{aligned} T_{101} T_0 T_{1012} &= (\xi_b^2 \xi_c + \xi_c) T_{01012} + \xi_b \xi_c T_{0102} \\ &\quad + \xi_b^2 T_{1012} + \xi_b T_{012} + \xi_b T_{102} + T_{02} \\ T_{101} T_2 T_{1012} &= T_{10121012} \\ T_{101} T_{1012} &= \xi_b^2 T_{210102} + \xi_b T_{0102} + \xi_c T_{1012} + \xi_b T_{12} + T_2 \end{aligned}$$

One can check that the conditions of Corollary 5.10 are satisfied. Hence $T_b C_d T_u = T_{bdu} \pmod{\mathcal{H}_{<0}}, \forall b \in B, u \in U$, i.e Proposition 5.7 in this case holds.

For the rest of “easy cases”, Proposition 5.7 follows by similar easy calculations. The main details are arranged in Appendix A.

5.11 The complicated cases: $(\tilde{C}_2, 1, \text{vii})$, $(\tilde{G}_2, 1, \text{i-iv})$, $(\tilde{G}_2, 2, \text{i-ii})$. For these cases, Proposition 5.7 does not hold; we have to return to Assumption 4.1(iv). This is one of reasons why the computations are complicated.

Cases $(\tilde{C}_2, 1, \text{vii})$. $a > c, a + c < 2b$. $B = \{e, 2, 12, 012\}$, $d = 101$, $U = U(2101)$.

$$\begin{aligned} T_{012}T_{10}T_{2101} &= T_{0212010} + \xi_b T_{02121010} \\ T_{012}T_{01}T_{2101} &= T_{010121201} \\ T_{012}T_0T_{2101} &= \xi_a T_{0120101} + \xi_b \xi_c T_{1010} + \xi_b T_{101} + \xi_c T_{010} + T_{10} \\ T_{012}T_1T_{2101} &= \xi_b T_{0121201} + T_{021201} \\ T_{012}T_{2101} &= \xi_a T_{012101} + \xi_b T_{1010} + \xi_c T_{01} + T_1 \end{aligned}$$

Using Lemma 5.9 and the equation (5.1), we have

$$(5.3) \quad T_{012}C_{101}T_{2101v} \equiv T_{012102101v} + T_{02121010v} - q^c T_{0121201v} \quad \forall v \in U.$$

Note that $1212 < 02121010v$ and $1212 < 0121201v$ and hence $1212 < \mathbf{c}$ and $1212 < \mathbf{c}$. One can check that $T_{02121010v} \in \mathcal{H}_{<0} + \mathcal{H}_{<\mathbf{c}}$ (which is easy) and $T_{0121201v} \in \mathcal{H}_{<0} + \mathcal{H}_{<\mathbf{c}}$ (using the condition $c < a$ and $c < b$). And hence

$$T_{012}C_{101}T_{2101v} \doteq T_{012102101v} \quad \forall v \in U.$$

Now one cannot conclude Assumption 4.1(iv) like the “easy cases”, since we cannot use Lemma 5.8 in this case. But we only need to do more similar calculations:

$$\begin{aligned} T_{12}C_{101}T_{2101v} &\equiv T_{12102101v} + T_{2121010v} - q^c T_{121201v}. \\ T_2C_{101}T_{2101v} &\equiv T_{2102101v}. \\ T_{012}C_{101}T_{210} &\equiv T_{01210210} - q^c T_{12120}. \\ T_{012}C_{101}T_{21} &\equiv T_{0121021} - q^c T_{01212}. \\ T_{12}C_{101}T_{21} &\equiv T_{121021} - q^c T_{1212}. \end{aligned}$$

Using these calculations, one can conclude Assumption 4.1(iv):

$$T_b C_{101} T_u \doteq T_{b101u} \quad \forall b \in B, u \in U.$$

The computations for the rest “complicated cases” are similar and are arranged in Appendix B.

6 Decomposition formula and conjectures P1-P15

In this section, we will see that the decomposition formula will be useful for the conjectures P1-P15. One can find our formulation $(P1)_{\leq \mathbf{c}} - (P15)_{\leq \mathbf{c}}$ in 2.8

Theorem 6.1 *Assume that*

- (a) **4.1** holds for the two-sided cells \mathbf{c} ;
- (b) $\mathbf{a}(d) = \deg h_{d,d,d}$ for any $d \in D$;
- (c) $(P1)_{<\mathbf{c}}, (P4)_{\leq \mathbf{c}}, (P8')_{\leq \mathbf{c}}$ holds.

Then

- (i) $(P1)_{\leq \mathbf{c}}$ holds.
- (ii) $\mathcal{D}_{\mathbf{c}} := \mathbf{c} \cap \mathcal{D}$ is the set $\{bdb^{-1} \mid d \in D, b \in B_d\}$.
- (iii) Every left cell and right cell in \mathbf{c} contains a unique element in \mathcal{D} .
- (iv) Let $q \in \mathcal{D}_{\mathbf{c}}$. Then $\gamma_{x,y,q} \neq 0$ if and only if $x = y^{-1}$, $y \sim_L q$. And in this case we have $\gamma_{x,y,q} = n_q = \pm 1$.

Proof. By Theorem 4.7, $C_{bd}C_{du} = h_{d,d,d}C_{bdu} \mod \mathcal{H}_{<\mathbf{c}}$. Hence we can write

$$(6.1) \quad C_{bd}C_{du} = h_{d,d,d}C_{bdu} + \sum_{z < \mathbf{c}} h_{bd,du,z}C_z.$$

Applying the map τ (see 2.6) to (6.1), we get

$$(6.2) \quad \delta_{bd,(du)^{-1}} = h_{d,d,d}P_{e,bdu} + \sum_{z < \mathbf{c}} h_{bd,du,z}P_{e,z} \mod \mathcal{A}_{<0}$$

where $\delta_{x,y} = 1$ if $x = y$, otherwise $\delta_{x,y} = 0$. By $(P1)_{<\mathbf{c}}$, we have $\mathbf{a}(z) \leq \Delta(z)$ for all $z < \mathbf{c}$. Hence

$$\sum_{z < \mathbf{c}} h_{bd,du,z}P_{e,z} = \sum_{z < \mathbf{c}} \gamma_{bd,du,z^{-1}}n_z \mod \mathcal{A}_{<0}.$$

By $(P8')_{\leq \mathbf{c}}$, we have $\gamma_{bd,du,z^{-1}} = 0$ since du and z are in different left cells.

Therefore

$$(6.3) \quad \sum_{z < \mathbf{c}} h_{bd,du,z}P_{e,z} = 0 \mod \mathcal{A}_{<0}.$$

Then equation (6.2) becomes

$$(6.4) \quad h_{d,d,d}P_{e,bdu} = \delta_{b,u^{-1}} \mod \mathcal{A}_{<0}.$$

So $\Delta(bdu) = -\deg P_{e,bdu} \geq \deg h_{d,d,d}$. By the hypothesis (b) and $(P4)_{\leq \mathbf{c}}$, $\deg h_{d,d,d} = \mathbf{a}(d) = \mathbf{a}(bdu)$. Then we get (i):

$$\mathbf{a}(bdu) \leq \Delta(bdu).$$

From (6.4), we can see also that $bdu \in \mathcal{D}$ if and only if $b = u^{-1}$. Then (ii) follows.

(iii) follows from Corollary 4.8 and (ii).

(iv). For $y \in \mathbf{c}$, we write Θ_y the left cell containing y . Denote by q_y the unique element of \mathcal{D} in Θ_y (see (iii)). For any $x, y \in \mathbf{c}$, consider the equation

$$(6.5) \quad \tau(C_x C_y) = \sum_{z \in \Theta_y} h_{x,y,z}P_{e,z} + \sum_{z < \mathbf{c}} h_{x,y,z}P_{e,z}.$$

For any $z \in \Theta_y$ with $z \neq q_y$, we have $\Delta(z) > \mathbf{a}(z)$ by (iii), and hence $h_{x,y,z}P_{e,z} \in \mathcal{A}_{<0}$. On the other hand, for all $z < \mathbf{c}$, we have $\deg h_{x,y,z}P_{e,z} < 0$ by the same reason as (6.3). Therefore, by taking constant terms in (6.5), we get

$$(6.6) \quad \delta_{x,y^{-1}} = \gamma_{x,y,q_y} n_{q_y}.$$

Note that n_{q_y} is a nonzero integer.

If $\gamma_{x,y,q} \neq 0$, then $y \sim_L q^{-1} = q$ by $(P8')_{\leq \mathbf{c}}$, i.e. $q = q_y$. Then we get $x = y^{-1}$ by (6.6). Conversely, if $x = y^{-1}$, $y \sim_L q^{-1} = q$, then $q = q_y$ and by (6.6) again we get $\gamma_{x,y,q} = \pm 1 = n_q$. This completes the proof of (iv). \square

Theorem 6.2 *Let \mathcal{E} be a free $\mathcal{A} \otimes \mathcal{A}$ -module with basis $\{\mathcal{E}_w \mid w \in \mathbf{c}\}$. Define a left \mathcal{H} -module structure on \mathcal{E} by*

$$C_x \mathcal{E}_w = \sum_{z \in \mathbf{c}} (h_{x,w,z} \otimes 1) \mathcal{E}_z \quad \text{for all } x \in W, w \in \mathbf{c}$$

and a right \mathcal{H} -module structure

$$\mathcal{E}_w C_y = \sum_{z \in \mathbf{c}} (1 \otimes h_{w,y,z}) \mathcal{E}_z \quad \text{for all } x \in W, w \in \mathbf{c}.$$

Assume that Assumption 4.1 holds for the two-sided cells \mathbf{c} and $\mathbf{a}(d) = \deg h_{d,d,d}$ for any $d \in D$. Then \mathcal{E} is an \mathcal{H} -bimodule, i.e the left and right module structures commute.

Proof. In this proof we will abbreviate $h'_{x,y,z} = h_{x,y,z} \otimes 1 \in \mathcal{A} \otimes \mathcal{A}$ and $h''_{x,y,z} = 1 \otimes h_{x,y,z} \in \mathcal{A} \otimes \mathcal{A}$. We first prove the following equation.

$$(6.7) \quad (C_{u^{-1}d} \mathcal{E}_d) C_{du'} = C_{u^{-1}d} (\mathcal{E}_d C_{du'}) \quad \text{for } d \in D, u, u' \in U_d.$$

By decomposition formula, we can see

$$C_{u^{-1}d} C_d \doteq E_{u^{-1}} C_d C_d \doteq h_{d,d,d} E_{u^{-1}} C_d \doteq h_{d,d,d} C_{u^{-1}d}$$

and hence $C_{u^{-1}d} \mathcal{E}_d = h'_{d,d,d} \mathcal{E}_{u^{-1}d}$. Similarly, we have $\mathcal{E}_d C_{du'} = h''_{d,d,d} \mathcal{E}_{du'}$.

Since $C_{u^{-1}d} C_{du'} \doteq h_{d,d,d} E_{u^{-1}} C_d F_{u'}$, $E_{u^{-1}} C_d F_{u'}$ is bar-invariant in $\mathcal{H}/\mathcal{H}_{<\mathbf{c}}$. So there exists bar invariant $b_z \in \mathcal{A}$ such that $E_u C_d F_{u'} \doteq \sum_{z \in \mathbf{c}} b_z C_z$. Hence $h_{u^{-1}d,du',z} = b_z h_{d,d,d}$, $\forall z \in \mathbf{c}$. By P4 and (b) in the hypothesis, $\mathbf{a}(z) = \mathbf{a}(d) = \deg(h_{d,d,d})$. Then we have $b_z \gamma_{d,d,d} = \gamma_{ud,du'^{-1},z^{-1}} \in \mathbb{Z}$. So

$$\mathcal{E}_{u^{-1}d} C_{du'} = \frac{h''_{d,d,d}}{\gamma_{d,d,d}} \sum_{z \in \mathbf{c}} \gamma_{ud,du'^{-1},z^{-1}} \mathcal{E}_z.$$

So

$$\begin{aligned} (C_{u^{-1}d} \mathcal{E}_d) C_{du'} &= h'_{d,d,d} \mathcal{E}_{u^{-1}d} C_{du'} \\ &= \frac{h'_{d,d,d} h''_{d,d,d}}{\gamma_{d,d,d}} \sum_{z \in \mathbf{c}} \gamma_{u^{-1}d,du',z^{-1}} \mathcal{E}_z \end{aligned}$$

Similarly, we have

$$C_{u^{-1}d}(\mathcal{E}_d C_{du'}) = \frac{h'_{d,d,d} h''_{d,d,d}}{\gamma_{d,d,d}} \sum_{z \in \mathbf{c}} \gamma_{u^{-1}d, du', z} \mathcal{E}_z.$$

Then equation (6.7) follows.

Let $x, y \in W$, $b \in B_d$, $u \in U_d$. By decomposition formula we have $C_{bd} \mathcal{E}_{du} = h'_{d,d,d} \mathcal{E}_{bdu}$.

$$\begin{aligned} (C_x \mathcal{E}_{bdu}) C_y &= \frac{1}{h'_{d,d,d}} (C_x (C_{bd} \mathcal{E}_{du})) C_y \\ &= \frac{1}{h'_{d,d,d}} ((C_x C_{bd}) \mathcal{E}_{du}) C_y \end{aligned}$$

(Since $(dU_d)^{-1} = \{v^{-1}d \mid v \in U_d\}$ is a left cell, we have $C_x C_{bd} = \sum_{v \in U_d} h_{x,bd,v^{-1}d} C_{v^{-1}d}$.)

$$\begin{aligned} &= \frac{1}{h'_{d,d,d}} \left(\sum_{v \in U_d} h'_{x,bd,v^{-1}d} C_{v^{-1}d} \mathcal{E}_{du} \right) C_y \\ &= \frac{1}{h'_{d,d,d} h''_{d,d,d}} \left(\sum_{v \in U_d} h'_{x,bd,v^{-1}d} C_{v^{-1}d} (\mathcal{E}_d C_{du}) \right) C_y \end{aligned}$$

by (6.7)

$$\begin{aligned} &= \frac{1}{h'_{d,d,d} h''_{d,d,d}} \left(\sum_{v \in U_d} h'_{x,bd,v^{-1}d} (C_{v^{-1}d} \mathcal{E}_d) C_{du} \right) C_y \\ &= \frac{h'_{d,d,d}}{h'_{d,d,d} h''_{d,d,d}} \left(\sum_{v \in U_d} h'_{x,bd,v^{-1}d} (\mathcal{E}_{v^{-1}d} C_{du}) \right) C_y \\ &= \frac{h'_{d,d,d}}{h'_{d,d,d} h''_{d,d,d}} \sum_{v \in U_d} h'_{x,bd,v^{-1}d} \mathcal{E}_{v^{-1}d} \left(\sum_{v' \in U_d} h''_{du,y,dv'} C_{dv'} \right) \\ &= \frac{1}{h'_{d,d,d} h''_{d,d,d}} \sum_{\substack{v \in U_d \\ v' \in U_d}} h'_{x,bd,v^{-1}d} h''_{du,y,dv'} (C_{v^{-1}d} \mathcal{E}_d) C_{dv'} \end{aligned}$$

Similarly,

$$C_x (\mathcal{E}_{bdu} C_y) = \frac{1}{h'_{d,d,d} h''_{d,d,d}} \sum_{\substack{v \in U_d \\ v' \in U_d}} h'_{x,bd,v^{-1}d} h''_{du,y,dv'} C_{v^{-1}d} (\mathcal{E}_d C_{dv'}).$$

By (6.7) again, we have $(C_x \mathcal{E}_{bdu}) C_y = C_x (\mathcal{E}_{bdu} C_y)$. Then the theorem follows. \square

Theorem 6.3 *Let W be an affine Weyl group of type \tilde{C}_2 or \tilde{G}_2 . If $P4$, $P8$ hold and $\mathbf{a}(d) = \deg h_{d,d,d}$ for all $d \in D$, then $P1$ - $P15$ hold.*

Proof. First $P14$ is obvious from the partition of two-sided cell of Guilhot [Gui10, Gui08], see Section 3. Recall that Assumption 4.1 holds for the affine

Weyl group of type \tilde{C}_2 or \tilde{G}_2 , see Section 5. Then we can apply Theorem 6.2 and we get immediately P15.

By induction, using Theorem 6.1 we obtain P1, P2, P3, P5, P6, P13 immediately.

By P4 and the assumption that $\mathbf{a}(d) = \deg h_{d,d,d}$ for all $d \in D$, we can determine the \mathbf{a} -values on each two-sided cell. By explicit computations of $\deg h_{d,d,d}$ (see Section 5), we can conclude that $\mathbf{a}(\mathbf{c}') < \mathbf{a}(\mathbf{c})$ for $\mathbf{c}' < \mathbf{c}$. Then we get P11.

Using P11 and [Gui10, Rem.6.7], we get P9, P10. And P3, P4, P8' imply P12, see [Lus03, §14.12]; P2, P3, P4, P5 imply P7, see [Lus03, §14.7]; P7 implies P8, see [Lus03, §14.8]. This completes the proof. \square

Remark 6.4 *It is an interesting question to compute the \mathbf{a} -function for the affine Weyl groups of type \tilde{C}_2 or \tilde{G}_2 . If the \mathbf{a} -values are determined, then the assumptions in Theorem 6.3 will be clear except P8'.*

A Computations for easy cases.

Case $(\tilde{C}_2, \mathbf{1}, \text{ii})$. $0 < a - c < 2b$. $B = \{e, , 2, 12, 012\}$, $d = 1010$, $U = U(2101)$.

$$\begin{aligned} T_{012}T_{101}T_{2101} &= T_{0121012101} \\ T_{012}T_{010}T_{2101} &= T_{1010212010} \\ T_{012}T_{10}T_{2101} &= \xi_b T_{01212010} + T_{0212010} \\ T_{012}T_{01}T_{2101} &= T_{01021201} \\ T_{012}T_0T_{2101} &= \xi_a T_{0102101} + \xi_b \xi_c T_{1010} + \xi_b T_{101} + \xi_c T_{010} + T_{10} \\ T_{012}T_1T_{2101} &= \xi_b T_{0121201} + T_{021021} \\ T_{012}T_{2101} &= \xi_a T_{012101} + \xi_b T_{1010} + \xi_c T_{01} + T_1. \end{aligned}$$

Case $(\tilde{C}_2, \mathbf{1}, \text{iii})$. $|a - c| < b$, $a + c > b$. $B = \{e, 1, 21, 01\}$, $d = 02$, $U = U(102)$.

$$\begin{aligned} T_{21}T_2T_{102} &= \xi_a T_{12120} + T_{1210} \\ T_{21}T_0T_{102} &= T_{210102} \\ T_{21}T_{102} &= \xi_b T_{2102} + \xi_a T_{02} + T_0. \end{aligned}$$

Case $(\tilde{C}_2, \mathbf{1}, \text{iv})$. $a + c < b$. $B = \{e, 0, 2, 02\}$, $d = 1$, $U = U(021)$.

$$T_{02}T_{021} = \xi_a \xi_c T_{021} + \xi_c T_{01} + \xi_a T_{21} + T_1$$

Case $(\tilde{C}_2, \mathbf{1}, \text{v})$. $a - c > b$. $B = \{e, 0, 10, 010\}$, $d = 212$, $U = U(012)$.

$$\begin{aligned} T_{010}T_{12}T_{012012} &= \xi_c T_{1010212012} + T_{101212012} \\ T_{010}T_{21}T_{012012} &= \xi_c T_{0121010212} + T_{012101212} \\ T_{010}T_1T_{012012} &= (\xi_b \xi_c^2 + \xi_b \xi_c) T_{101212} + \xi_c^2 T_{010212} + \xi_c T_{10212} \\ &\quad + \xi_b T_{0101212} + \xi_c T_{01212} + T_{1212} \end{aligned}$$

$$T_{010}T_2T_{012012} = \xi_c T_{010210212} + T_{01212012}$$

$$T_{010}T_{012012} = \xi_c^2 + \xi_c T_{101212} + \xi_b T_{012012} + \xi_c T_{2012} + T_{212}$$

Case $(\tilde{C}_2, \mathbf{1}, \mathbf{vi})$. $a > c, a + c > 2b$. $B = \{e, 1, 01, 101\}$, $d = 2$, $U = U(1012)$.

$$T_{101}T_{1012} = \xi_b^2 T_{10102} + \xi_b T_{0120} + \xi_c T_{1012} + \xi_b T_{12} + T_2$$

Case $(\tilde{C}_2, \mathbf{2}, \mathbf{i-vi})$. The verification is very easy.

Case $(\tilde{C}_2, \mathbf{3}, \mathbf{i})$. $a - c = 2b$. $B_1 = \{e, 1, 01\}$, $d_1 = 02$, $U_1 = U(1012)$, $B_2 = \{e\}$.

$$T_{01}T_0T_{101} = \xi_b \xi_c T_{1010} + \xi_b T_{101} + \xi_c T_{010} + T_{10}$$

$$T_{01}T_2T_{101} = T_{012101}$$

$$T_{01}T_{101} = \xi_b T_{1010} + \xi_c T_{01} + T_1.$$

Case $(\tilde{C}_2, \mathbf{3}, \mathbf{ii})$. $a + c = b$. $B_1 = \{e, 0, 2\}$, $d_1 = 1$, $U_1 = U(021)$, $B_2 = \{e\}$.

$$T_0T_{02} = \xi_c T_{02} + T_2$$

Case $(\tilde{C}_2, \mathbf{3}, \mathbf{iii})$. $a - c = b$. $B_1 = \{e, 1, 01\}$, $d_1 = 02$, $U_1 = U(102)$, $B_2 = \{e\}$.

$$T_{01}C_{02}T_{102102} = T_{0102102102} + q^{-c}T_{012102102} + q^{-a}\xi_c T_{10102120} + q^{-a}T_{1012120}$$

$$+ q^{-a-c}\xi_b T_{0102120} + q^{-a-c}\xi_c T_{02102} + q^{-a-c}T_{2102}$$

Case $(\tilde{C}_2, \mathbf{3}, \mathbf{iv})$. $a > c, a + c = 2b$. $B_1 = \{e, 1, 01\}$, $d_1 = 2$, $U_1 = U(1012)$, $B_2 = \{e\}$.

$$T_{01}T_{1012} = \xi_b T_{01012} + \xi_c T_{012} + T_{12}.$$

Case $(\tilde{C}_2, \mathbf{3}, \mathbf{v})$. $a = c, a > b$. $B_1 = B_2 = \{e, 1\}$, $d_1 = 2$, $U_1 = U(1012)$, $d_2 = 0$, $U_2 = U(1210)$.

$$T_1C_2T_{1012} = T_{121012} + q^{-a}T_1^2T_{012}.$$

Case $(\tilde{C}_2, \mathbf{3}, \mathbf{vi})$. $a = c, a < b$. $B_1 = \{e, 2\}$, $d_1 = 101$, $U_1 = U(2101)$, $B_1 = \{e, 0\}$, $d_2 = 121$, $U_2 = U(0121)$.

$$T_2T_{10}T_{2101} = T_{2120101}$$

$$T_2T_{01}T_{2101} = T_{0212101}$$

$$T_2T_0T_{2101} = T_2^2T_{1010}$$

$$T_2T_1T_{2101} = T_{121201}$$

$$T_2T_{2101} = T_2^2T_{101}$$

Case $(\tilde{C}_2, \mathbf{4}, \mathbf{i-ii})$. $\forall d \in D$, $B_d = \{e\}$ and hence there is nothing need to verify.

Case $(\tilde{C}_2, \mathbf{4}, \mathbf{iii})$. We only need to verify that $T_0C_1T_{210} \equiv T_{01210}$, which is easy.

Case $(\tilde{G}_2, \mathbf{1}, \mathbf{v})$. $2a > 3b, a < 2b$. $c = \{e, 1, 21, 121, 0121\}02\{e, 1, 12, 121, 1210\}$.

$$T_{0121}T_0T_{1210} = T_{012101210}$$

$$T_{0121}T_2T_{1210} = \xi_b T_{01212120} + T_{0212120}$$

$$T_{0121}T_{1210} = \xi_b T_{0121210} + \xi_a T_{01210} + \xi_b T_{010} + \xi_b T_0 + 1$$

Case $(\tilde{G}_2, \mathbf{1}, \mathbf{vi})$. $a < b$. $c = \{e, 0\}12121\{e, 0\}$.

$$T_0 C_{12121} T_0 \equiv T_{0121210} + (q^{-3b}(\eta_{2a} - 1) - q^{-3b}\xi_a T_2) T_2^2 \equiv T_{0121210}.$$

Case $(\tilde{G}_2, \mathbf{3}, \mathbf{i})$. $\forall d \in D$, $B_d = \{e\}$ and hence there is nothing need to verify.

Case $(\tilde{G}_2, \mathbf{3}, \mathbf{ii})$. We only need to compute $T_2 C_1 T_{212} \equiv T_{21212}$.

Case $(\tilde{G}_2, \mathbf{3}, \mathbf{iii})$. $\forall d \in D$, $B_d = \{e\}$ and hence there is nothing need to verify.

Case $(\tilde{G}_2, \mathbf{3}, \mathbf{iv})$. $T_{01} C_2 T_{10} \equiv T_{01210}$.

B Computations for complicated cases

Cases $(\tilde{G}_2, \mathbf{1}, \mathbf{i})$. $2a > 3b$. $B = \{e, 0, 10, 210, 1210, 01210\}$, $d = 21212$, $U = U(01212)$.

$$\begin{aligned} T_{01210} T_{2121} T_{0121201212} &= T_{0121021210121201212} \\ T_{01210} T_{1212} T_{0121201212} &= T_{0121012120121201212} \\ T_{01210} T_{212} T_{0121201212} &= T_{012102120121201212} \\ T_{01210} T_{121} T_{0121201212} &= \xi_b T_{01201210121201212} + \xi_a \xi_b T_{12012121012121} \\ &\quad + \xi_a T_{1210121201212} + \xi_b T_{1012121012121} + T_{102121012121} \\ T_{01210} T_{12} T_{0121201212} &= \xi_b T_{0121201212101212} + \xi_a \xi_b T_{1021212101212} \\ &\quad + \xi_a T_{102121201212} + \xi_b T_{101212101212} + T_{10212101212} \\ T_{01210} T_{21} T_{0121201212} &= \xi_b T_{0121210121201212} + \xi_a \xi_b T_{0212121012121} \\ &\quad + \xi_a T_{021212012121} + \xi_b T_{012121021212} + T_{02121021212} \\ T_{01210} T_2 T_{0121201212} &= \xi_b T_{012102121201212} + \xi_a \xi_b T_{021212101212} \\ &\quad + \xi_a T_{02121201212} + \xi_b T_{01212101212} + T_{0212101212} \\ T_{01210} T_1 T_{0121201212} &= \xi_b^2 T_{01210121201212} + \xi_b T_{0120121201212} \\ &\quad + \xi_b T_{0121021201212} + \xi_a T_{01021201212} + \xi_b^2 T_{01021212} \\ &\quad + \xi_b T_{1021212} + \xi_b T_{0121212} + T_{212121} \\ T_{01210} T_{0121201212} &= \xi_b T_{01210121201212} + \xi_b T_{012121201212} \\ &\quad + \xi_a T_{0121201212} + \xi_b T_{01201212} + \xi_b T_{021212} + T_{21212}. \end{aligned}$$

Since $C_{21212} = C_2 T_1 T_2 T_1 C_2 - q^{-a} \xi_b C_2 T_1 C_2 + q^{-2a}(\eta_{2b} - 1)C_2$, we have

$$\begin{aligned} T_{01210} C_{21212} T_{0121201212u} &\equiv T_{01210212120121201212u} + q^{2b-a} T_{02121201212u} \\ &\quad + q^{3b-a} T_{012121201212u} - q^{2b-a} (T_{1012121201212u} + T_{0121212012121u}) \end{aligned}$$

Further computations shows that

$$\begin{aligned} T_{01210} C_{21212} T_{012120v} &\equiv T_{0121021212012120v} + q^{2b-a} T_{0212120v} \\ &\quad + q^{3b-a} T_{01212120v} - q^{2b-a} (T_{101212120v} + T_{012121012v}) \\ T_{01210} C_{21212} T_{01212} &\equiv T_{012102121201212} + q^{2b-a} T_{021212} \\ &\quad + q^{3b-a} T_{0121212} - q^{2b-a} T_{10121212} \end{aligned}$$

$$\begin{aligned}
T_{01210}C_{21212}T_{01210} &\equiv T_{012102121201210} \\
T_{1210}C_{21212}T_{0121201212u} &\equiv T_{1210212120121201212u} + q^{2b-a}T_{2121201212u} \\
&\quad + q^{3b-a}T_{21212101212u} - q^{2b-a}T_{121212012121u}. \\
T_{1210}C_{21212}T_{01212w} &\equiv T_{12102121201212w} + q^{2b-a}T_{21212w} + q^{3b-a}T_{212121w} \\
T_{210}C_{21212}T_{01212z} &\equiv T_{2102121201212v} + q^{2b-a}T_{121212z} \\
T_{10}C_{21212}T_{0121201212u} &\equiv T_{10212120121201212u}
\end{aligned}$$

where $v \in \{e, 1, 12, 121\}$, $w \in \{e, 0, 01, 012, 0121\}$, $z \in \{e, 0, 01, 012, 0121, 01212\}$. Using condition $2a > 3b$, one can check that for any $u \in U$,

$$\begin{aligned}
q^{2b-a}T_{121212u} &\equiv q^{2b-a}C_{121212}T_u \\
q^{2b-a}T_{21212u} + q^{3b-a}T_{121212u} &\equiv q^{2b-a}C_{121212}T_u. \\
q^{2b-a}T_{0121212u} &\equiv q^{2b-a}T_0C_{121212}T_u \\
q^{2b-a}T_{10121212u} &\equiv q^{2b-a}T_{10}C_{121212}T_u \\
q^{2b-a}T_{021212u} + q^{3b-a}T_{0121212u} &\equiv q^{3b-a}T_0C_{121212}T_u. \\
(\text{When } 2b > a) \quad q^{2b-a}T_{012121012v} &\equiv q^{2b-a}T_{01212}C_{101}T_{2v}
\end{aligned}$$

Then we can conclude that $T_bC_dT_u \dot{=} T_{bdu}$, for any $b \in B, u \in U$.

Case $(\tilde{G}_2, \mathbf{1}, \text{ii})$. $2a < 3b$. $B = \{e, 1, 21, 121, 0121, 2121\}$, $d = 02$, $U = U(12120)$.

$$\begin{aligned}
T_{0121}T_0T_{12120} &= T_{0121012120} \\
T_{0121}T_2T_{12120} &= \underline{\xi_a\xi_bT_{01212120} + \xi_aT_{0212120} + \xi_bT_{0121210} + T_{021210}} \\
T_{0121}T_{12120} &= T_2 + \xi_bT_{02} + \xi_bT_{1012} + \xi_aT_{012120} + \xi_bT_{01212120}
\end{aligned}$$

Then for all $u \in U$ we have

$$T_{0121}C_{02}T_{12120u} \equiv T_{01210212102u} + q^{a-b}T_{0212120u} + q^aT_{01212120u} + T_{0121210u}.$$

Further computations shows that

$$\begin{aligned}
T_{0121}C_{02}T_{1212v} &\equiv T_{0121021212v} + q^{a-b}T_{021212v} + q^aT_{0121212v} + T_{012121v}. \\
T_{0121}C_{02}T_{1210} &\equiv T_{0121021210}. \\
T_{121}C_{02}T_{1212v} &\equiv T_{121021212v} + q^{a-b}T_{21212v} + q^aT_{121212v} + T_{12121v}. \\
T_{21}C_{02}T_{1212v} &\equiv T_{21021212v} + q^{a-b}T_{121212v}. \\
T_1C_{02}T_{1212v} &\equiv T_{1021212v}.
\end{aligned}$$

where v is the elements such that $1212v \in U$. By

$$\begin{aligned}
q^aC_{0121212v} &\equiv q^{a-b}T_{021212v} + q^aT_{0121212v} + T_{012121v}, \\
q^aC_{121212v} &\equiv q^{a-b}T_{21212v} + q^aT_{121212v} + T_{12121v}, \\
q^{a-b}C_{121212v} &\equiv q^{a-b}T_{121212v},
\end{aligned}$$

for all $b = e, 0121, 121, 21, 1, \forall u \in U$, we have $T_bC_dT_u \dot{=} T_{bdu}$ (note that $121212 < \mathbf{c}$). It remains to verify Assumption 4.1(iv) for $b = 2121$. Since

$$T_{2121}T_0T_{12120} = T_{2121012120}$$

$$\begin{aligned}
T_{2121}T_2T_{12120} &= T_{1210} + \xi_a T_{12120} + \xi_a T_{21210} \\
&\quad + \xi_a \xi_b T_{121210} + \xi_a^2 T_{212120} + \xi_a^2 \xi_b T_{1212120} \\
T_{2121}T_{12120} &= T_0 + \xi_a T_{02} + \xi_b T_{2120} \\
&\quad + \xi_b T_{121210} + \xi_a T_{212120} + \xi_a \xi_b T_{1212120}.
\end{aligned}$$

we have for all $u \in U$,

$$\begin{aligned}
T_{2121}C_{02}T_{12120u} &\equiv T_{21210212120u} + T_{1212120u} + (q^{2a} - 2 - q^{2a-2b})T_{1212120u} \\
&\quad + q^a T_{121210u} + q^{2a-b} T_{212120u} + q^{a-b} T_{21210} + q^{a-b} T_{12120u}.
\end{aligned}$$

Further computations show that

$$\begin{aligned}
T_{2121}C_{02}T_{1212v} &\equiv T_{2121021212v} + T_{121212v} + (q^{2a} - 2 - q^{2a-2b})T_{121212v} \\
&\quad + q^a T_{12121v} + q^{2a-b} T_{21212v} + q^{a-b} T_{2121v} + q^{a-b} T_{1212v}.
\end{aligned}$$

where v is an element such that $1212v \in U$. Now by

$$\begin{aligned}
q^{2a}C_{121212}T_v &\equiv q^{2a}T_{121212v} + q^a T_{12121v} + q^{2a-b} T_{21212v} \\
&\quad + q^{a-b} T_{2121v} + q^{a-b} T_{1212v} \\
(2 + q^{2a-2b})C_{121212}T_v &\equiv (2 + q^{2a-2b})T_{121212v},
\end{aligned}$$

we can see Assumption 4.1(iv) holds for $b = 2121, u \geq_D 1212$.

Case ($\tilde{G}_2, \mathbf{1}, \mathbf{iii}$). $a > 2b$. $B = \{e, 1, 01, 21, 121, 0121\}$, $d = 02$, $U = U(102)$.

$$\begin{aligned}
T_{0121}T_0T_{102102} &= T_{120121210} + \xi_b T_{1012121012} \\
T_{0121}T_2T_{102102} &= T_{201212012} + \xi_b T_{0212121012} \\
T_{0121}T_{102102} &= T_{12} + \xi_b T_{120} + \xi_b T_{012} + \xi_b^2 T_{1012} \\
&\quad + \xi_a T_{1012120} + \xi_b T_{012121012}.
\end{aligned}$$

Then $T_{0121}C_{02}T_{102102u} \equiv T_{012102102102u} + T_{0121212012u}$, $\forall u \in U$. Further computations show that

$$\begin{aligned}
T_{0121}C_{02}T_{10210v} &\equiv T_{01210210210v} + T_{012121201v} \\
T_{0121}C_{02}T_{10212} &\equiv T_{01210210212} \\
T_{121}C_{02}T_{10210v} &\equiv T_{1210210210v} + T_{12121201v} \\
T_{21}C_{02}T_{10210v} &\equiv T_{210210210v} \\
T_{01}C_{02}T_{10210v} &\equiv T_{010210210v}
\end{aligned}$$

So $T_b C_d T_u \equiv T_{bdu}$, $\forall b \in B, u \in U$.

Case ($\tilde{G}_2, \mathbf{1}, \mathbf{iv}$). $a < 2b$. $B = \{e, 2, 12, 212, 1212, 01212\}$. $d = 101$. $U = U(210)$.

$$\begin{aligned}
T_{01212}T_{10}T_{210210} &= T_{02121201210} + \xi_b T_{012121201210} \\
T_{01212}T_{01}T_{210210} &= T_{0121201210210} \\
T_{01212}T_1T_{210210} &= \xi_a(\xi_b^2 + 1)T_{012121201} + \xi_a \xi_b T_{02121201} + \\
&\quad \xi_b^2 T_{01212101} + \xi_b T_{0212101} + \xi_b T_{0121201} + T_{021201} \\
T_{01212}T_0T_{210210} &= T_{10121210} + \xi_b T_{101212101} + \xi_a T_{01212101210}
\end{aligned}$$

$$\begin{aligned} T_{01212}T_{210210} &= T_1 + \xi_b T_{10} + \xi_b T_{01} + \xi_b^2 T_{101} + \xi_a T_{101210} \\ &\quad + \xi_a T_{20121201} + \xi_b T_{01212101} + \xi_a \xi_b T_{201212101} \end{aligned}$$

Then for all $u \in U$,

$$\begin{aligned} T_{01212}C_{101}T_{210210u} &\equiv T_{01212101210210u} + T_{012121201210u} + T_{01212101u} \\ &\quad + q^{a-b}T_{02121201u} + q^a T_{012121201u}. \end{aligned}$$

Further computations show that

$$\begin{aligned} T_{01212}C_{101}T_{2102z} &\equiv T_{012121012102z} + q^{a-b}T_{20121210z} + T_{0212121012z}(z \in \{e, 1\}) \\ T_{01212}C_{101}T_{210} &\equiv T_{01212101210} + T_{021212101} \\ T_{1212}C_{101}T_{210210u} &\equiv T_{1212101210210u} + T_{12121201210u} + T_{1212101u} \\ &\quad + q^{a-b}T_{2121201u} + q^a T_{12121201u} \\ T_{1212}C_{101}T_{2102w} &\equiv T_{12121012102w} + q^{a-b}T_{1212120w}(w \in \{e, 1\}) \\ T_{212}C_{101}T_{210210u} &\equiv T_{212101210210u} + q^{a-b}T_{12121201u}(u \in U) \\ T_{212}C_{101}T_{21021} &\equiv T_{21210121021} \\ T_{12}C_{101}T_{210210u} &\equiv T_{12101210210u}(u \in U). \end{aligned}$$

Since

$$\begin{aligned} q^a C_{012121201u} &\equiv q^{a-b}T_{02121201u} + q^a T_{012121201u}, \\ q^a C_{12121201u} &\equiv q^{a-b}T_{2121201u} + q^a T_{12121201u}, \\ q^{a-b}C_{121212} &\equiv q^{a-b}T_{121212}, \end{aligned}$$

we have $T_b C_d T_u \equiv T_{bdu}, \forall b \in B, d \in U$.

Case($\tilde{G}_2, \mathbf{2}, \mathbf{i}$). It is part of computations of Case ($\tilde{G}_2, \mathbf{1}, \mathbf{ii}$). The detail is omitted.

Case($\tilde{G}_2, \mathbf{2}, \mathbf{ii}$). It is part of computations of Case ($\tilde{G}_2, \mathbf{1}, \mathbf{iii}$). The detail is omitted.

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